G12-MDM-Skewness

October 26, 2019

1 Skewness

Let's consider the following two datasets x and y (ignore the python code if you are not familiar with the language).

[2]:
```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
import scipy.stats as stats

fn='https://evermeet.cx/~user055/Dragon/Lessons/G12MDM/AutomobileDataEDA.csv'

df = pd.read_csv(fn)
df.head(10)             #let's show the data on the first 10 cars of the list



tb=df[['price','city-mpg']].head(10)
tb.columns=['x','y']
tb.loc[4,'x']=57398
xs=tb.sort_values(by='x')
ys=tb.sort_values(by='y')
#print(xs['x'].values[2]-(xs['x'].values[2]-xs['x'].values[3])/2 )
xs
print("The table of data values:\n",tb)
x=tb['x'].values
y=tb['y'].values
tb.describe()
```

```
The table of data values:
          x   y
0  13495.0  21
1  16500.0  21
2  16500.0  19
3  13950.0  24
4  57398.0  18
5  15250.0  19
6  17710.0  19
7  18920.0  19
```

```
8  23875.0  17
9  16430.0  23
```

[2]:

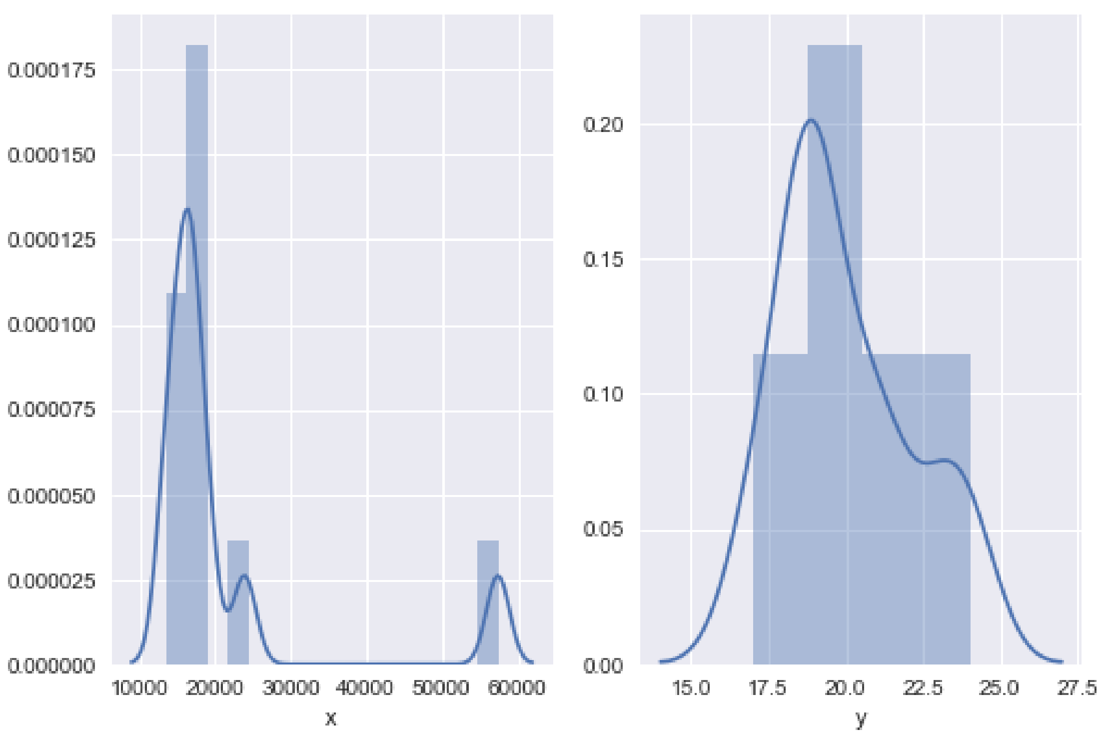
	x	y
count	10.00000	10.000000
mean	21002.80000	20.000000
std	13117.86473	2.211083
min	13495.00000	17.000000
25%	15545.00000	19.000000
50%	16500.00000	19.000000
75%	18617.50000	21.000000
max	57398.00000	24.000000

1.1 The histogram and its skewness

Let's look at the histograms of x and y

[73]:
```
import seaborn as sns

ax1=plt.subplot(121)
axx=sns.distplot(x,hist=True,axlabel='x',ax=ax1)
ax2=plt.subplot(122)
axy=sns.distplot(y,hist=True,axlabel='y',ax=ax2)
```



We can easily see that neither of those two histograms are symmetric around one single peak. Also, they show a tail towards the right (its more clear for the case of x). We say the histograms are *skewed*.

How can we measure that skewness

Consider for a moment the histogram for x. Having a long tail means that many values are "small" but a few are much larger than the mean.

If the histogram were symmetric, for each value of x at a give distance above its mean, one would get another value below the mean but at the same distance from it.

Hence, in order to quantify that skewnes, we may want to look at the differences wrt the mean.

Case 1: Simple differences $(x - \overline{x})$ Let's consider the differences $x - \overline{x}$. What is its average? What about for y?

[86]:
```
xdif = xs[['x']].copy(deep=True)
xdif['$x-\overline x$']=x-x.mean()
print('Mean x:',x.mean())
xdif
```

```
Mean x: 21002.8
```

[86]:

	x	$x-\overline x$
0	13495.0	-7507.8
3	13950.0	-4502.8
5	15250.0	-4502.8
9	16430.0	-7052.8
1	16500.0	36395.2
2	16500.0	-5752.8
6	17710.0	-3292.8
7	18920.0	-2082.8
8	23875.0	2872.2
4	57398.0	-4572.8

[87]:
```
xdif['$x-\overline x$'].mean()
```

[87]: `3.637978807091713e-13`

We see that the average value of $x - \overline{x}$ is less than 4 divided by 10 trillion! In Computer Science lingo, this is the same as saying it's zero!! You can easily convince yourself that the same happens for y.

Case 2: Differences squared $(x - \overline{x})^2$ What about the following values $(x - \overline{x})^2$? These are no longer zero -they will never be zero unless your measurements always get one and the same result!

In particular we get an average of this difference squared of 154870537.56 and 4.4, for x and y respectively.

[89]:
```
meanOfDifferencesSquared_x=((x-x.mean() )**2 ).mean()
meanOfDifferencesSquared_y=((y-y.mean() )**2 ).mean()
meanOfDifferencesSquared_x,meanOfDifferencesSquared_y
```

[89]: (154870537.55999994, 4.4)

However, as we are squaring the differences, we cannot distinguish between a value below the mean and one above the mean: *when squaring we loose the sign of that difference*. This means this quantity **cannot capture our intuition of skewness either**.

Case 3: Differences cubed$(x - \overline{x})^3$ Let's then look at the differences to the 3rd power, $(x - \overline{x})^3$.

As we know from math, the odd power of a negative number is also negative. Example: $(-3)^3 = -27$, but $(-3)^4 = +81$!

This time, then, we won't loose the information of the sign of these differences, in other words, **these differences are able to capture the left or right bias of the data**!

As the mean of a set of numbers is always *very sensitive to extreme values and outliers*, these differences will be dominated by the tail of the histogram. This means, **if the tail is towards the right, the skewness will be positive, and if the tail is towards the left, the skewness will be negative**.

Example Let's consider a simple example. Consider the list of four numbers $a = [0, 1, 2, 3]$. Clearly, the mean is 1.5 and for each value below this mean there is another above it at the same distance. Hence the skewnness will be zero. **Check it by calculating it yourself!**

Let's now repeatedly swap that 0 by a 10, 100, 1000 and check in each case what happens with the skewness. Note: We use here another notation for the mean, namely the angled brackets $< x >$.

[123]:
```python
print("Positive Skewness:\nOutlier\tMean\tMean((x-<x>)^3)\tskewness (standard;␣
↪see text below)")
for l in [0,10, 100,1000]:
    a = np.array([l,1,2,3])
    print(str(l)+":\t",a.mean(),"\t",((a-a.mean())**3).mean(),"\t\t",stats.
↪skew(a))
```

```
Positive Skewness:
Outlier Mean    Mean((x-<x>)^3) skewness (standard; see text below)
0:       1.5     0.0             0.0
10:      4.0     45.0            1.0182337649086284
100:     26.5    88200.0                 1.1537390557978817
1000:    251.5   93188250.0              1.154691263747523
```

Let's repeat the experiment with values of 0, -10, -100, -1000.

[124]:
```python
print("Negative skewness:\nOutlier\tMean\tMean((x-<x>)^3)\tskewness (standard;␣
↪see text below)")
for l in [0,-10, -100,-1000]:
    a = np.array([l,1,2,3])
    print(str(l)+":\t",a.mean(),"\t",((a-a.mean())**3).mean(),"\t\t",stats.
↪skew(a))
```

```
Negative skewness:
Outlier Mean    Mean((x-<x>)^3) skewness (standard; see text below)
0:       1.5     0.0             0.0
```

```
-10:     -1.0    -157.5                    -1.092148056772224
-100:    -23.5   -99450.0                  -1.1538129615124053
-1000:   -248.5          -94313250.0               -1.154691337648407
```

Let's see now what we get for our datasets x and y.

1.1.1 Skewness unscaled

[39]:
```
((y-y.mean() )**3 ).mean()
```

[39]:
```
5.4
```

[40]:
```
((x-x.mean() )**3 ).mean()
```

[40]:
```
4694581371666.743
```

Wow! The mean value of those differences cubed is astronomically higher for x than for y. **However, this doesn't correspond to what we *see when looking at their histograms*!!**

Something is odd about using just the differences cubed.

It seems reasonable to expect the result for x to be that much higher because the scale of the values of x is about 2000 times that of y. And $8000^3 = 8 \cdot 10^6$, i.e., when cubing the result becomes 8 million times larger! Yet, this back of the envelop calculation doesn't completely explain the even larger difference we got for both datasets. Why?

What really matters in this calculation are not the values themselves, but the scale of their differences with respect to the mean. For y the largest differences is 4, while for x is about 40000. **That is, the scale difference that really matters is $10000 = 10^4$. When cubing this we get 10^{12}, a trillion times larger for x than for y as our calculations show!!**

We need thus a way to properly compare this values without being affected by the intrinsic scale of each data set.!

One way is by dividing those differences by their mean.

1.1.2 Skewness scaling by the mean

[31]:
```
((y-y.mean() )**3 ).mean()/y.mean()**3
```

[31]:
```
0.000675
```

[125]:
```
((x-x.mean() )**3 ).mean()/x.mean()**3
```

[125]:
```
0.5067167735408883
```

When dividing those differences by the mean we obtain values easier to grasp. Yet, the result for x is still 1000 times that for y!

This is still not quite what we would have expected, isn't?

Look again at the histograms. Both show a right bias, sure. But would you say that of x is thousand times larger than that of y!? Clearly, no!

The reason for such unintuitive disparity of skewness values is that we are comparing the differences against the wrong number: Instead of comparing against the mean value, **we should compared against the mean difference with respect to the mean!**

But we have seen above that the mean difference with respect to the mean is always zero!

Hence, we need to compare against the standard deviation!

1.1.3 Skewness scaling by the standard deviation

[37]:
```
((y-y.mean() )**3 ).mean()/y.std()**3
```

[37]:
```
0.585079316127977
```

[38]:
```
stats.skew(x)
```

[38]:
```
2.43581142085585
```

This now feels like something we could agree upon: the bias in x is 4 times that of y.
This is the definition of the skewness we will be using from now on, namely

$$\text{Skewness} \equiv \frac{1}{N}\sum_{i=1}^{i=N}\left(\frac{(x_i - \overline{x})}{S_x}\right)^3$$

where S_x is another notation for the standard deviation of x.
In case of doubt we will called this expression of the skewness the *standard skewness*.